Recursive sequences

1. The set of positive real numbers $a_1, a_2, \dots, a_{2n+1}$ is one such that $a_k - 2a_{k+1} + a_{k+2} \ge 0$, $k = 1, 2, \dots, (2n-1)$. Show that $\dfrac{a_1 + a_3 + \dots + a_{2n+1}}{n+1} \ge \dfrac{a_2 + a_4 + \dots + a_{2n}}{n}$.

Also prove that the equality holds if the numbers are in an arithmetic progression.

Deduce that if $0 < t < 1$, $\dfrac{1-t^{n+1}}{n+1} > \dfrac{1-t^n}{n}\sqrt{t}$.

2. The sequence $a_1, a_2, a_3, \dots$ is defined as $a_1 = 3$, $a_{n+1} = \dfrac{a_n^{\ 2} + 5}{2a_n}$, $(n > 0)$.

Prove that $$0 < a_{n+1} - \sqrt{5} < \frac{\left(3-\sqrt{5}\right)^{2^n}}{\left(2\sqrt{5}\right)^{2^n-1}} < 6\left(\frac{2}{11}\right)^{2^n}.$$

3. **(i)** The terms of a sequence $y_1, y_2, y_3, \dots$ satisfy the relation $y_k = Ay_{k-1} + B$ $(k \ge 2)$, where A and B are constants independent of k and $A \ne 1$.

Guess an expression for y_k $(k \ge 2)$ in terms of y_1, A, B and k, and prove it.

(ii) The terms of a sequence $x_0, x_1, x_2, \dots$ satisfy the relation $x_k = (a + b)\, x_{k-1} - ab\, x_{k-2}$ $(k \ge 2)$, where a, b are non-zero constants independent of k and $a \ne b$.

(a) Express $x_k - ax_{k-1}$ $(k \ge 2)$ in terms of $(x_1 - a\, x_0)$, b and k.

(b) Using (i), or otherwise, express x_k $(k \ge 2)$ in terms of x_0, x_1, a, b and k.

(iii) If the terms of the sequence $x_0, x_1, x_2, \dots$ satisfy the relation $x_k = \dfrac{1}{3}x_{k-1} + \dfrac{2}{3}x_{k-2}$ $(k \ge 2)$,

express $\lim\limits_{k\to\infty} x_k$ in terms of x_0 and x_1.

4. Prove that if n is a positive integer, then $\left(\sqrt{3}+1\right)^n = a_n\sqrt{3} + b_n$ for unique integers a_n, b_n. Furthermore, prove that **(i)** $a_{n+2} = 2(a^{n+1} + a_n)$, $b_{n+2} = 2(b^{n+1} + b_n)$;

(ii) $\left(\sqrt{3}-1\right)^n = (-1)^{n-1}\left(a_n\sqrt{3} - b_n\right)$;

(iii) $3a_n^{\ 2} - b_n^{\ 2} = (-1)^{n-1}2^n$.

5. The two sequences of positive integers $a_1, a_2, \dots, a_n, \dots$; $b_1, b_2, \dots, b_n, \dots$ satisfy the following conditions: $a_1 = b_1 = 1$ and $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ for all positive integers n.

(i) Prove that for each positive integer n, $a_n \ge n, b_n \ge n$ and $a_n^{\ 2} - 2b_n^{\ 2} = (-1)^n$.

(ii) Reduce from **(i)** that $\dfrac{a_n}{b_n} < \sqrt{2}$, if n is odd , $\dfrac{a_n}{b_n} > \sqrt{2}$ if n is even and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \sqrt{2}$.

(iii) Express $\dfrac{a_{n+1}}{b_{n+1}}$ in terms of $\dfrac{a_n}{b_n}$ and show that $\left|\dfrac{a_{n+1}}{b_{n+1}} - \sqrt{2}\right| < \left|\dfrac{a_n}{b_n} - \sqrt{2}\right|$.

6. Let x_n, y_n be numbers defined by $x_{n+1} = \sqrt{x_n y_n}$, $y_{n+1} = \dfrac{x_n + y_n}{2}$,

$x_1 = a > 0, y_1 = b > 0$ and $a > b$. Show that $x_n \le y_n$ for $n > 1$.